A Finite Element Solution for Saint-Venant Torsion

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A solution of the Saint-Venant Torsion Problem is developed using the finite element method. Difficulties with multiply-connected regions are completely avoided by the introduction of a force variable that explicitly defines any traction-free boundary. A direct matrix relation is established between the forces and displacements such that the solution for any section is obtained in terms of the displacements at discrete points. In addition to discrete or distributed inhomogeneity the method is developed to treat any anisotropic effect that can be characterized by arbitrarily oriented orthotropy of individual elements. Matrix relations are derived and discussed for triangular and square elements. Applications to a solid and a hollow isotropic square section are presented and the results are shown to converge to the exact elasticity solution. A computer program using arbitrary quadrilateral elements is briefly described in an application to the analysis of cooled gas turbine blades.

Nomenclature‡

A = area
C = contour
G = isotropic torsional modulus
$G_{xz'}, G_{yz'}$ = torsional modulii on principal axes of orthotropy
I_p = polar moment of inertia of area in x-y plane
J = torsional stiffness of section[1]
l,m = direction cosines of outward normal to a given contour
s,n = tangential and normal coordinates respective to a given contour
S = arc length
T = torque on ends of prism
u,v = displacements in the plane of the section respective to x,y directions
w = out-of-plane warping displacement in the z direction
w_i = nodal displacements
x,y = Cartesian coordinates in a plane normal to the axis of rotation
x',y' = local principal axes of orthotropy
z = Cartesian coordinate along axis of rotation
Z_i = nodal forces
α = angle of twist per unit length[1]
γ_{xz}, γ_{yz} = shearing strains
δ = variational operator as applied to virtual displacements
θ = angle between local x' axis and the x axis
ξ = longitudinal traction on lateral boundary surface (Fig. 1)
τ_{xz}, τ_{yz} = shearing stresses
ϕ = warping function, defined by $w = \alpha\phi$[1,2]

Introduction and Formulation

THE theory of Saint-Venant Torsion offers an exact solution for any isotropic and homogeneous prism whose ends are loaded in torsion by a specific shear stress distribution and are allowed to warp freely.[1,2] When the ends of the prism are restrained from warping or the torque is applied in any other manner, the Saint-Venant Solution is accurate for all sections sufficiently removed from the ends for the Saint-Venant Principle to apply. The theory may be equally extended to certain types of anisotropy[3-5] and inhomogeneity[6] wherein no variation of properties occurs along the axis of torsion.

A large class of engineering problems involves torsion of nonprismatic bars with end restraint. In such cases no exact solutions exist, but useful approximations may be obtained if the Saint-Venant Solution is known for the sections involved. If the Saint-Venant warping function is known, the problems of restrained warping may be treated as in Refs. 2 and 7. For curved, twisted, or tapered bars, various strength-of-materials solutions may be applied if the Saint-Venant characteristics have been previously determined.[8-10] An accurate solution to the Saint-Venant problem is thus of general importance in the analysis of structures.

Closed form solutions for Saint-Venant torsion exist only for a few geometrically regular sections due to the mathematical difficulties involved in satisfying the governing equations. For this reason several approximate techniques have been evolved for this problem. These techniques, however, are limited to thin sections and can be inaccurate in an undefined manner.[7,8,11] Direct numerical techniques as applied to the exact theory offer a general approach to any type of section and are, therefore, of great interest and utility.

The most common formulation of the torsion problem is in terms of the Prandtl stress function.[1,2] This approach has been used in connection with finite difference[1,11] and finite element[12] techniques. The stress function formulation, however, is difficult to apply to complex sections and does not directly provide warping displacements for the consideration of the restrained warping problem. The direct stress technique of Ref. 13 is easier to apply to complex sections but does not directly obtain the warping displacements.

The torsion problem may also be described by a displacement formulation the solution of which is of advantage for further application to the restrained warping problem. The method of Ref. 14 obtains a displacement formulation by a finite element technique that combines the potential energy of all elements adjacent to a given node. This energy is then minimized to obtain a relation between the displacement at a given node and that of the surrounding nodes.

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‡ All rotations, angles, and contour integrals are considered positive according to the right-hand rule.

The present paper presents a finite element representation of the torsion problem by a direct stiffness method similar to that employed in other continuum problems.[15–17] By the principle of virtual work, a stiffness relation for an element is established between the nodal forces (Z_i) and nodal displacements (w_i). The elements adjacent to a given node are assembled by establishing equilibrium of the forces between elements ($\Sigma Z_i = 0$) and continuity of displacements.

A finite element of a section in Saint-Venant torsion described by the contour C and length L whose faces are normal to the axis of rotation is shown in Fig. 1. In general the C-L surface will be loaded by surface tractions ξ, which vary along the contour, are constant in the z direction, and act parallel to the z axis. As in the finite element analysis of other problems,[15–17] these surface tractions are replaced by a system of forces Z_i per unit length in the z direction acting at discrete points P_i along the contour C and parallel to the z axis or

$$\xi_i = \lim_{S_i \to 0} Z_i/S_i L$$

where S_i is the arc length between adjacent points and ξ_i is the distributed traction on S_i. The ξ_i and Z_i system of forces are equivalent if the virtual work done by each system is equal and hence

$$\sum_{i=1}^{i=k} Z_i \delta w_i = \oint_C \xi \delta w \, dS = \sum_{i=1}^{i=k} \int_{S_i} \xi \delta w \, dS \tag{1}$$

For an element of orthotropic material with a principal axis of orthotropy parallel to the z axis, the stress-strain relations are[5]

$$\tau_{xz} = G_{xz}\gamma_{xz} + H\gamma_{yz}\tau_{yz} = H\gamma_{xz} + G_{yz}\gamma_{yz} \tag{2}$$

where

$$\begin{aligned} G_{xz} &= G_{xz'} \cos^2\theta + G_{yz'} \sin^2\theta \\ G_{yz} &= G_{xz'} \sin^2\theta + G_{yz'} \cos^2\theta \\ H &= (G_{xz'} - G_{yz'}) \sin\theta \cos\theta \end{aligned} \tag{3}$$

and θ is the angle between the "x" principal axis of orthotropy and the "x" coordinate axis. It should be noted that θ can be different from element to element. By the principle of potential energy, the virtual work of the Z_i forces is equal to variation of the strain energy and hence

$$\sum_{i=1}^{i=k} (\delta w_i) Z_i = \iint_A [(\delta\gamma_{xz})\tau_{xz} + (\delta\gamma_{yz})\tau_{yz}] dA$$

or

$$\sum_{i=1}^{i=k} (\delta w_i) Z_i = \alpha^2 \iint_A \left\{ \delta\left(\frac{\partial\phi}{\partial x}\right)\left[G_{xz}\left(\frac{\partial\phi}{\partial x} - y\right) + H\left(\frac{\partial\phi}{\partial y} + x\right)\right] + \delta\left(\frac{\partial\phi}{\partial y}\right)\left[H\left(\frac{\partial\phi}{\partial x} - y\right) + G_{yz}\left(\frac{\partial\phi}{\partial y} + x\right)\right]\right\} dA \tag{4}$$

since

$$\gamma_{xz} = \alpha(\partial\phi/\partial x - y), \; \gamma_{yz} = \alpha(\partial\phi/\partial y + x)$$

$$\delta\gamma_{xz} = \alpha\delta(\partial\phi/\partial x), \; \delta\gamma_{yz} = \alpha\delta(\partial\phi/\partial y)$$

The virtual work relation (4) defines the Z_i shear forces in terms of the displacement field within any finite region. If permissible displacements are assumed within the element, the area integration can be performed and an algebraic relation may be obtained between the forces and displacements. Having this relation for each element, all elements may then be combined (as in Ref. 15) to describe the total section or a portion of a symmetric section.

The procedure for deriving specific finite element relations from the virtual work Eq. (4) is quite similar to that employed in plane stress and plane strain problems.[15–17] For illustration, a triangular element will be considered though various other representations are possible. The more elementary details of manipulation are omitted since they are easily accessible in the references and in standard texts.

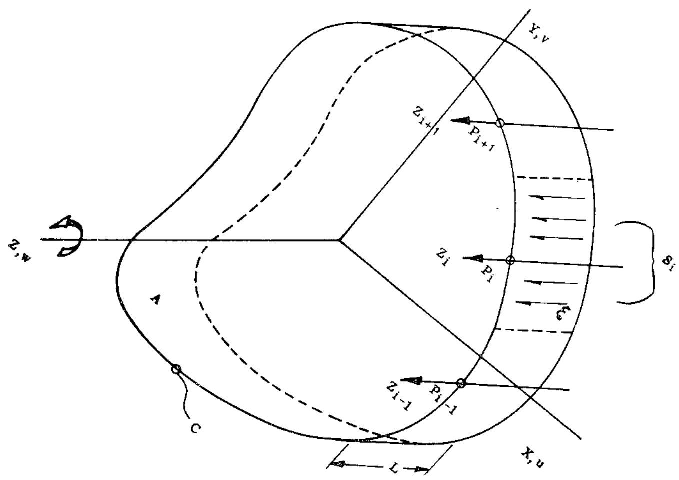

Fig. 1 A finite element showing discrete and distributed shear tractions.

Consider a triangular element bounded by straight lines connecting the points x_1,y_1; x_2,y_2; x_3,y_3, numbered in a counterclockwise manner. Assume that if the element is sufficiently small the warping function may be approximated by the linear relation

$$w/\alpha = \phi = B_1 + B_2 x + B_3 y \tag{5}$$

After expressing the B_i coefficients in Eq. (5) in terms of the corner displacements w_i, Eq. (5) may be substituted into Eq. (4) to yield the matrix relations

$$\begin{Bmatrix} Z_1 \\ Z_2 \\ Z_3 \end{Bmatrix} = \frac{1}{4A}\begin{bmatrix} k_{11} & k_{12} & k_{13} \\ & k_{22} & k_{23} \\ & & k_{33} \end{bmatrix}\begin{Bmatrix} w_1 \\ w_2 \\ w_3 \end{Bmatrix} + \frac{\alpha}{2}\begin{Bmatrix} C_1 \\ C_2 \\ C_3 \end{Bmatrix} \tag{6}$$

where

$$\begin{aligned} k_{11} &= G_{xz}Y_{23}^2 + G_{yz}X_{23}^2 + 2HX_{32}Y_{23} \\ k_{12} &= G_{xz}Y_{31}Y_{23} + G_{yz}X_{31}X_{23} + H(Y_{31}X_{32} + X_{31}Y_{32}) \\ k_{13} &= G_{xz}Y_{21}Y_{32} + G_{yz}X_{21}X_{32} + H(Y_{21}X_{23} + X_{21}Y_{23}) \\ k_{22} &= G_{xz}Y_{31}^2 + G_{yz}X_{31}^2 + 2HY_{13}X_{31} \\ k_{23} &= G_{xz}Y_{31}Y_{12} + G_{yz}X_{31}X_{12} + H(Y_{21}X_{31} + X_{21}Y_{31}) \\ k_{33} &= G_{xz}Y_{21}^2 + G_{yz}X_{21}^2 + 2HY_{12}X_{21} \\ C_1 &= G_{xz}\bar{y}Y_{32} + G_{yz}\bar{x}X_{32} + H(\bar{x}Y_{23} + \bar{y}X_{23}) \\ C_2 &= G_{xz}\bar{y}Y_{13} + G_{yz}\bar{x}X_{13} + H(\bar{x}Y_{31} + \bar{y}X_{31}) \\ C_3 &= G_{xz}\bar{y}Y_{21} + G_{yz}\bar{x}X_{21} + H(\bar{x}Y_{12} + \bar{y}X_{12}) \end{aligned}$$

$\bar{x},\bar{y}$ are the coordinates of the center of gravity of the triangular area and

$$X_{ij} = x_i - x_j, \; Y_{ij} = y_i - y_j$$

For a rectangular element with sides parallel to the coordinate axes, the warping function may be approximated by

$$w/\alpha = \phi = B_1 + B_2 x + B_3 y + B_4 xy \tag{7}$$

Use of assumption (7) as compared to that of (5) will be discussed in the next article.

The stiffness relation for any finite element is of the form

$$\{Z\} = [K]\{w\} + \{C\}$$

The K matrix represents the stiffness contribution when warping is present while the C vector is the contribution from section rotation alone (Coulomb theory). Superposition of

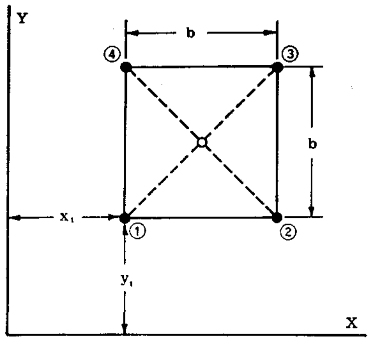

Fig. 2 A square element with sides parallel to the coordinate axes.

all elements in a section leads to a similar relation for the entire section. Setting the boundary forces to zero on free surfaces enforces the zero traction condition. Setting the boundary displacements to zero on lines of symmetry allows analysis of partial sections. For a complete section the specification of zero displacement at one point in the region removes the singularity of possible rigid body translation along the z axis. The warping displacements for any problem may be obtained by solution of the linear simultaneous equations of the assembled matrix.

Once the warping displacements have been determined the stresses may be evaluated from the warping derivatives by differentiation of the warping assumption (5). A numerical integration over the section then provides the torsional stiffness. Further treatment of the initial warping as in Ref. 18 determines the center of twist and the principal warping.

Numerical Examples

The matrix relations for an isotropic and homogeneous square element (Fig. 2 solid lines) may be derived using assumption (7):

$$\begin{Bmatrix} Z_1 \\ Z_2 \\ Z_3 \\ Z_4 \end{Bmatrix} = \frac{G}{48}\begin{bmatrix} 32 & -8 & -16 & -8 \\ & 32 & -8 & -16 \\ & & 32 & -8 \\ & & & 32 \end{bmatrix}\begin{Bmatrix} w_1 \\ w_2 \\ w_3 \\ w_4 \end{Bmatrix} + \frac{\alpha G b}{2}\begin{Bmatrix} y_1 - x_1 \\ -(y_1 + x_1 + b) \\ x_1 - y_1 \\ (y_1 + x_1 + b) \end{Bmatrix} \quad (8)$$

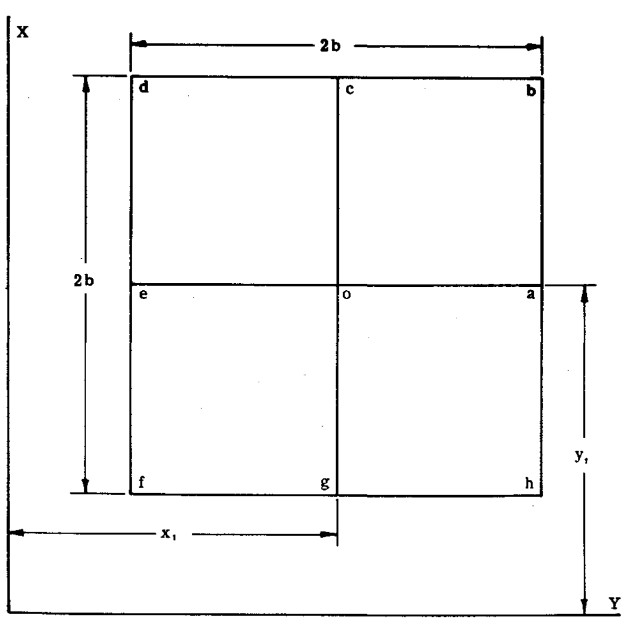

Fig. 3 Assembly of four square elements.

Table 1 Comparison of torsional stiffness of a square section using finite element and finite difference methods

	$J/a^4 = 2.2500$ (exact)[1]		
No. Elements	Matrix (10)	Matrix (11)	Finite Difference[1]
16	2.367	2.3254	2.125
64	2.280	2.2706	2.2175
256		2.2548	
1024		2.2506	

Comparable relations may be obtained by combining four triangular elements (Fig. 2 dotted lines) based on assumption (5). The internal point may be eliminated since its force superposition is complete;

$$\begin{Bmatrix} Z_1 \\ Z_2 \\ Z_3 \\ Z_4 \end{Bmatrix} = \frac{G}{48}\begin{bmatrix} 36 & -12 & -12 & -12 \\ & 36 & -12 & -12 \\ & & 36 & -12 \\ & & & 36 \end{bmatrix}\begin{Bmatrix} w_1 \\ w_2 \\ w_3 \\ w_4 \end{Bmatrix} + \frac{\alpha G b}{2}\begin{Bmatrix} y_1 - x_1 \\ -(y_1 + x_1 + b) \\ x_1 - y_1 \\ (y_1 + x_1 + b) \end{Bmatrix} \quad (9)$$

The major difference between (8) and (9) is in the relative magnitude of the off-diagonal terms in the K matrices. Considering the influence of the force Z_1 in Eqs. (8), it may be seen that the coefficient of the warping deflection of point 3 is twice as large as that at 2 or 4. This is physically unreasonable since point 3 is further from the point of load application than 2 or 4 and a lesser coefficient should be expected. The situation is improved in Eqs. (9) since points 2, 3, and 4 have equal deflection coefficients.

The behavior of matrices (8) and (9) may be further compared to the finite difference method. In the assembly of four elements (Fig. 3), the force superposition at the central point 0 is complete so that its deflection in terms of those at adjacent points may be determined from the equation $Z_0 = 0$;

$$8w_0 - (w_a + w_b + w_c + w_d + w_e + w_f + w_g + w_h) = 0 \text{ for matrix (8)} \quad (10)$$

$$12w_0 - 2(w_a + w_c + w_e + w_g) - (w_b + w_d + w_f + w_h) = 0 \text{ for matrix (9)} \quad (11)$$

In this case (10) gives equal weight to all the adjacent points whereas (11) gives twice as much influence to those nearer the central point. By the finite difference method, the deflection at point 0 would be[1]

$$4w_0 - (w_a + w_c + w_e + w_g) = 0$$

The torsional stiffness of any section may be found by integration;

$$T = \alpha J = \int_A (x\tau_{yz} - y\tau_{xz})dA$$

For the square element of Fig. 2, this may be found for assumption (7) by direct integration;

$$J = \frac{Gb}{2}[(y_1 - x_1)(w_1 - w_3) + (x_1 + y_1 + b)(w_4 - w_2)] + GI_p$$

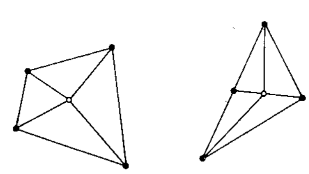

Fig. 4 Examples of quadrilateral elements.

Table 2 Effect of no. of elements on the convergence of torsional stiffness and maximum shearing stress, for a hollow square section

No. Elements	J/a^4	$\tau_{max}/G\alpha a$
12	2.1024	1.3196
48	2.0784	1.3184
200	2.0688	1.3162
Exact	2.0592	1.3162

For the four-element section of Fig. 3, the total stiffness may be found as the sum of the element stiffnesses;

$$J = (Gb/2)[(y_f - x_f)(w_f - w_b) + 2(b + x_f)(w_c - w_g) + 2(b + y_f)(w_e - w_a) + (2b + x_f + y_f)(w_d - w_h)] + GI_p$$

where I_p is the total polar moment of inertia for all four elements. Similar relations may be found from the triangular element resulting from assumption (5). In either case, the torsional stiffness of any section may be easily found from the displacements.

For comparison, the torsional stiffness of a solid square section ($2a \times 2a$) was computed using matrix (8) and matrix (9). These results are presented in Table 1 along with the finite difference values from Ref. 1.

As an example of a multiply connected region consider the torsion of a square section of sides $2a$ with a central square hole of sides a. The problem was solved using matrix (9) to analyze a quarter section of the square by equating the warping deflections to zero on the lines of symmetry. Results are given in Table 2 and compared to the exact values of Ref. 19.

The results for these two simple examples show that the finite element method of this paper converges quickly to the exact elasticity solution. The convergence of matrix (9) is equal to or better than that of matrix (8) or the finite difference method. It may then be concluded that the simple formulation for a triangular element, i.e., assumption (5), is adequate for this problem.

For irregular shapes, a more general element is required. In such cases a stiffness relation may be obtained for an arbitrary quadrilateral element by again combining four triangular elements (Fig. 4). This approach was incorporated into a computer program that is in common use to analyze, for example, complex regions such as the cooled turbine blade section shown in Fig. 5. The complete solution of this problem including stresses, torsional stiffness, location of the center of twist, and the principal warping deflections was accomplished in less than 30 sec on the Univac 1108.

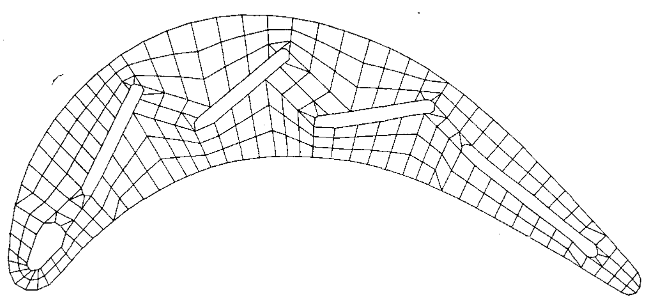

Fig. 5 Finite element representation of a turbine blade section.

Results from this program have been compared; wherever possible, with exact[1,2,6,11] solutions and accepted approximate (thin-section) techniques. In general, values for torsional stiffness may be obtained within 1% error using 100 quadrilateral elements in a complete section. Values for stresses depend highly on the size of the elements in areas of stress concentration. In such regions very accurate results may be obtained by analyzing a local section alone using displacement boundary conditions from a previous large mesh result.

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